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The representation theory of the icosahedral group

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Abstract. A study is made of the representation theory of the icosahedral group I by exploiting the natural embeddings of the latter in the rotation groups in three and four dimensions. In particular it is shown that each irreducible representation of I is the restriction of some irreducible representation of SO(4), and that this enables the irreducible representations of I to be symmetrized.

1. Introduction

In a recent paper (Backhouse and Gard 1974a), an account was given of the reduction of the symmetrized powers of the irreducible representations (reps) of the three-dimensional point groups. The work was incomplete in that it was not possible to treat fully the icosahedral group, I, using the methods developed therein. The difficulty is that I possesses some reps which are neither induced linear characters nor subduced from reps of SO(3). The aim of the present paper is to demonstrate that the symmetrization problem is nevertheless tractable for I by the use of some features of its character theory, which are also of interest in their own right.

There are at least three realizations of the abstract group behind I: the proper rotation group of the icosahedron (or dodecahedron) in three-dimensional Euclidean space E_3; the proper rotation group, P_5 say, of the regular pentrahedroid (regular four-simplex) in E_4; the alternating group on five letters, A_5. In this paper we exploit the two geometrical characterizations, but make little mention of the third. The reason for this, briefly, is that the isomorphism between I, P_5 and A_5, and more generally between P_n and A_n, has considerable value for the representation theory of A_5, generally A_n, especially their spin characters. However, it is the insight into the representation theory of I in which we are primarily interested here.

The structure of the paper is as follows. In § 2 we consider the relationship between the rep theory (both single- and double-valued) of I and that of the rank one Lie group SO(3), and in particular we perform the reduction of the subduced representations $D^j \downarrow I$, where D^j is the $(2j+1)$-dimensional rep of SO(3), for $j = 0, \frac{1}{2}, 1, \ldots$. The value of these reductions is enhanced by the introduction of an algebraic operation (denoted by $\sim$) on the character table of I which has the special property of interchanging the two two-dimensional reps, interchanging the two three-dimensional reps, but leaving the others invariant. By combining these results with the formulae for symmetrizing the reps of SU(2), the natural universal covering group of SO(3) (see Gard and Backhouse 1974), we are able to symmetrize the reps of I, with one exception. Unfortunately the single-valued four-dimensional rep of I does not fall nicely within this scheme; the best we can say is that it is the Kronecker product of the two two-dimensional reps, and hence may be symmetrized using the inner Kronecker product theory for symmetric groups.

Although this fills the lacuna in Backhouse and Gard (1974a), there is virtue in looking at the embedding of I in SO(4). Indeed, in § 3, we are able to tie together the results of § 2 and also find a better method of symmetrizing the awkward four-dimensional rep. We note in particular that every rep of I is the restriction of some rep of SO(4). The fact that I embeds in SO(4) is well known, but to our knowledge the discussion of their relative rep structures is new.

The icosahedral group is becoming of increasing importance in quantum physics and chemistry (see, for example, the recent paper of Boyle 1972 and the works cited therein). Although I is non-crystallographic in three dimensions, it is nevertheless true that in certain crystals there is a local icosahedral environment, and hence there are applications of I in crystal field theory. We also remark that four-dimensional orthogonal groups are well known in quantum chemistry (see for example Wulfman 1971).

2. The icosahedral group

It is well known that the three-dimensional proper symmetry group I of the icosahedron and its dual, the dodecahedron, for whose constructive details we refer to Cundy and Rollett (1961), is isomorphic to the alternating group A_5. I is a simple group of order 60 having its elements distributed into five conjugacy classes as follows: the trivial class is denoted by C_1; C_2 denotes the class containing 12 rotations through angles $2\pi/5$ and $8\pi/5$; C_3 denotes the class containing 12 rotations through angles $4\pi/5$ and $6\pi/5$; the class containing 15 two-fold rotations is denoted by C_4; the class containing 20 three-fold rotations is denoted by C_5. I has five single-valued reps and four double-valued reps, the characters of the latter vanishing on the class C_4 by Opechowski's theorem (see also Backhouse 1973). In table 1 of the appendix we reproduce the character table of I. Also table 3 contains, among other things, relationships between the reps of I and SO(3).

We find that the reps T_2, $E_{7/2}$, G are not directly related to reps of SO(3), yet there is some connection between T_2 and $T_1 = D^1 \downarrow I$ and between $E_{7/2}$ and $D^{1/2} \downarrow I$. We also note that $G = E_{1/2} \otimes E_{7/2}$. Our first goal is to explain these relationships in order that manipulations may be performed within the SO(3) context.

It is sometimes convenient to consider separately single- and double-valued characters, but there is some virtue now in viewing them as the full set of inequivalent simple characters of the double group I'. Then, considered as class functions on I', the simple characters generate, by taking finite integral sums and differences, an algebraic structure called the integral character ring of I', denoted by $C(I')$. Multiplication is also allowed, being distributive over addition, if reductions are performed using the Kronecker product formula. Now we notice that $(1+\sqrt{5})/2 = \omega+1+\omega^4$ and $(1-\sqrt{5})/2 = \omega^2+1+\omega^3$, where $\omega = \exp(2\pi i/5)$, hence the values of the characters of I' can be written in the ring obtained by adjoining fifth roots of unity to the integers. This ring possesses an automorphism, $\sim$, defined by

$$a_0+a_1\omega+a_2\omega^2+a_3\omega^3+a_4\omega^4 \overset{\sim}{\rightarrow} a_0+a_1\omega^2+a_2\omega^4+a_3\omega+a_4\omega^3$$

for integers a_i, $i = 0, 1, \ldots, 4$. Briefly, $\sim$ leaves the integers invariant but squares the fifth roots of unity. The important point for us is that $\sim$ interchanges $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, hence the ring automorphism induced by $\sim$ on $C(I')$ interchanges T_1 and T_2, interchanges $E_{1/2}$ and $E_{7/2}$, but leaves the other simple characters invariant. Furthermore, because $\sim$ preserves multiplication, we can write $\widetilde{\Gamma_1 \otimes \Gamma_2} = \tilde{\Gamma}_1 \otimes \tilde{\Gamma}_2$, for characters Γ_1, Γ_2 of I', and $(\tilde{\Gamma})^{[v]} = \widetilde{\Gamma^{[v]}}$, where $\Gamma^{[v]}$ is the $[v]$-symmetrized nth power of the character Γ corresponding to the rep $[v]$ of the symmetric group S_n. We add finally that the above results could more briefly be said to be consequences of the fact that $(1\pm\sqrt{5})/2$ are algebraically conjugate roots of the irreducible monic polynomial x^2-x-1.

The procedure to be followed for symmetrizing the reps of I is now clear—we are here reverting to the single group notion. If $\chi = \chi^j \downarrow I$ is a character of I, where χ^j is the character of the rep D^j of SO(3), then $\chi^{[v]} = (\chi^j)^{[v]} \downarrow I$, $(\chi^j)^{[v]}$ being reduced using the theory of Gard and Backhouse (1974). If χ, a character of I, is such that $\tilde{\chi} = \chi^j \downarrow I$, for some j, then $\chi^{[v]} = \widetilde{(\chi^j)^{[v]} \downarrow I}$. To symmetrize the exceptional rep $G \equiv E_{1/2} \otimes \tilde{E}_{1/2}$, we use the following theorem which is an easy consequence of the Frobenius formula (see, for example, p 331 of Weyl 1950).

Theorem. Let D_1, D_2 be representations of the group G, and let $[v]$ denote a rep of the symmetric group S_n. Then

$$(D_1 \otimes D_2)^{[v]} = \bigoplus_{[v_1],[v_2]} p([v];[v_1],[v_2]) D_1^{[v_1]} \otimes D_2^{[v_2]}, \tag{2.1}$$

where $[v_1], [v_2]$ are reps of S_n and $p([v];[v_1],[v_2])$ is the frequency of $[v]$ in the inner Kronecker product $[v_1] \otimes [v_2]$.

In the case under consideration, D_1, D_2 are two-dimensional, so it suffices to restrict the summation in (2.1) to those reps $[v_1], [v_2]$ with one- and two-rowed Young's diagrams. We refer the readers to Hamermesh (1964) for the known details of the Clebsch–Gordan theory of the symmetric groups.

In order to implement the relationship between I and SO(3) for the solution of the problem of symmetrizing the reps of I, we also need to know the reductions $D^j \downarrow I$ for all j. It turns out that there is a considerable regularity and fine structure which simplifies the solution. Using the character formula

$$\chi^j(\theta) = \frac{\sin(j+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \qquad \text{for } \theta \neq 0,$$

for the reps of SO(3), we find the following relations. First, for single-valued reps

$$\phi^{j+30} = \phi^j + (\text{Reg}), \tag{2.2}$$

where ϕ^j is the character of $D^j \downarrow I$, and (Reg) denotes the character of the regular rep of I, namely $A+3T_1+3T_2+4G+5H$;

$$\phi^j + \phi^{j'} = (\text{Reg}), \tag{2.3}$$

if $j+j' = 29$; $2\phi^7$, $\phi^9+\phi^5$, $\phi^{11}+\phi^3$, $\phi^{13}+\phi^1$ have a common value $\alpha = 2T_1+2T_2+2G+2H$: $\phi^8+\phi^6$, $\phi^{10}+\phi^4$, $\phi^{12}+\phi^2$, $\phi^{14}+\phi^0$ have a common value $\beta = A+T_1+T_2+2G+3H$. Note that $\alpha+\beta = (\text{Reg})$. It now suffices to calculate $\phi^0 = A$, $\phi^1 = T_1$, $\phi^2 = H$, $\phi^3 = T_2+G$, $\phi^4 = G+H$, $\phi^5 = T_1+T_2+H$, $\phi^6 = A+T_1+G+H$.

Secondly, for half-integer reps we have

$$\phi^{j+15} = \phi^{j} + \tfrac{1}{2}(\mathrm{Reg})', \tag{2.4}$$

where (Reg)′ denotes the double-valued regular character, namely $2E_{1/2}+2E_{7/2}+4G_{3/2}+6I_{5/2}$;

$$\phi^{j} + \phi^{j'} = \tfrac{1}{2}(\mathrm{Reg})', \tag{2.5}$$

if $j+j' = 14$; $\phi^{13/2}+\phi^{5/2}$, $\phi^{11/2}+\phi^{7/2}$ have a common value δ, where $\delta+\phi^{9/2} = \frac{1}{2}(\mathrm{Reg})'$. It now suffices to calculate $\phi^{1/2} = E_{1/2}$, $\phi^{3/2} = G_{3/2}$, $\phi^{5/2} = I_{5/2}$, $\phi^{7/2} = E_{7/2}+I_{5/2}$, $\phi^{9/2} = G_{3/2}+I_{5/2}$. These results fully extend the tabulations of Cohan (1958).

3. The pentahedroidal group

The regular figure which is the four-dimensional analogue of the regular tetrahedron and the equilateral triangle is called the regular pentahedroid. For example, the five points $A(4/\sqrt{5}, 0, 0, 0)$, $B(-1/\sqrt{5}, -1, 1, 1)$, $C(-1/\sqrt{5}, 1, -1, 1)$, $D(-1/\sqrt{5}, 1, 1, -1)$, $E(-1/\sqrt{5}, -1, -1, -1)$ form the vertices of such a figure, for it is easy to check that the vertices are equidistant from one another, and that angles between incident lines are equal. Note that any four of the vertices are those of a regular tetrahedron. We have arranged the vertices so that the proper symmetry group of BCDE is $\mathbf{1} \oplus \boldsymbol{T}$, as a subgroup of SO(4), where $\mathbf{1}$ acts as the identity on the first coordinate and $\boldsymbol{T}$ is the proper tetrahedral group, in a standard setting, acting on the last three coordinates. Now we may verify directly that the special orthogonal matrix M, given by

$$4M = \begin{pmatrix} -1 & -\sqrt{5} & -\sqrt{5} & -\sqrt{5} \\ -\sqrt{5} & -1 & -1 & 3 \\ \sqrt{5} & 1 & -3 & 1 \\ \sqrt{5} & -3 & 1 & 1 \end{pmatrix}, \tag{3.1}$$

permutes the vertices in the order $A \to B \to C \to D \to E \to A$, hence represents a cyclic operation of order 5. Together M and $\mathbf{1} \oplus \boldsymbol{T}$ (isomorphic to A_4) generate a group, $\boldsymbol{P}_5$ say, isomorphic to $\boldsymbol{A}_5$. It is clear that the adjunction of further symmetry operations, which can only be of an improper nature, increases the symmetry group to one isomorphic to $\boldsymbol{S}_5$. The latter is the maximum possible, which is proved using a special case of the well known result in E_n that there is at most one distance-preserving operation which maps any given $n+1$ distinct points onto any given $n+1$ distinct points. Thus we have explicitly embedded $\boldsymbol{A}_5(\boldsymbol{I})$ as $\boldsymbol{P}_5$ in SO(4) and also $\boldsymbol{S}_5(\boldsymbol{I}_\mathrm{h})$ in O(4). Finally, by prolongation of the edges of the pentahedroid, it is clear that it can be taken as the unit cell of a Bravais lattice and hence we verify that $\boldsymbol{P}_5$ is a crystallographic point group.

We now turn to the relationship between the rep theory of SO(4) and that of $\boldsymbol{I}$. There are two possible approaches to the rep theory of SO(4), which we briefly review here. First, we may regard SO(4) as the special case $n = 4$ of the group SO(n), whose rep theory is expounded, for example, in Boerner (1970). The conjugacy classes of SO(4) are labelled by two angles (ϕ, ψ) corresponding to the elements in a maximal torus of the form $E(\phi) \oplus E(\psi)$ where

$$E(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{3.2}$$

The reps of SO(4), denoted $D^4_{m_1m_2}$ in Boerner (1970), are labelled by pairs (m_1, m_2), $m_1 \geqslant |m_2|$, single-valued reps having m_1, m_2 both integral, and double-valued reps having m_1, m_2 both half-integral. The value of the character of $D^4_{m_1m_2}$ at the class (ϕ, ψ) is

$$\frac{\cos[(m_1+1)\phi+m_2\psi]-\cos[(m_1+1)\psi+m_2\phi]}{\cos\phi-\cos\psi}, \tag{3.3}$$

and the dimension is $(m_1+1)^2-m_2^2$.

In order to discuss the relationship between the reps of SO(4) and $\boldsymbol{I}$ it is necessary to specify the values of (ϕ, ψ) for the classes of the latter group. These values appear in table 2 in the appendix, and were determined with the aid of Hurley (1951). We note from table 3 that each rep of $\boldsymbol{I}$ is the restriction of some rep of SO(4), which means that operations on the reps of $\boldsymbol{I}$ may be performed on the corresponding reps of SO(4), provided that we are able to reduce the subduced reps $D^4_{m_1m_2} \downarrow \boldsymbol{I}$, for all (m_1, m_2). Unfortunately the sort of regularities which appeared when we restricted reps of SO(3) to $\boldsymbol{I}$ are now not so easy to describe using the above rep theory of SO(4). We do, however, observe that $\tilde{\chi}_{m_1m_2} = \chi_{m_1,-m_2}$, where $\chi_{m_1m_2}$ is the character of $D^4_{m_1m_2} \downarrow \boldsymbol{I}$.

However, we can overcome the difficulty by appealing to an alternative approach to the reps of SO(4). This depends on the result that the natural universal covering group of SO(4) is SU(2)$\times$SU(2). We recall the following from Talman (1960). Let

$$X = \begin{pmatrix} t+\mathrm{i}z & -y+\mathrm{i}x \\ y+\mathrm{i}x & t-\mathrm{i}z \end{pmatrix}, \tag{3.4}$$

be any member of SU(2). If A, B are further members of SU(2), then so is AXB^{-1}, which may be written uniquely in the form

$$\begin{pmatrix} t'+\mathrm{i}z' & -y'+\mathrm{i}x' \\ y'+\mathrm{i}x' & t'+\mathrm{i}z' \end{pmatrix}. \tag{3.5}$$

Then it is easy to check that $R:(x, y, z, t) \to (x', y', z', t')$ extends linearly to an SO(4) operation, and moreover that the map $(A, B) \to R$ of SU(2)$\times$SU(2) $\to$ SO(4) is 2–1, onto and a homomorphism. Also the maximal torus in SO(4) corresponds to choosing A, B in the form $\pm\mathrm{diag}(\exp \mathrm{i}\alpha/2, \exp -\mathrm{i}\alpha/2)$. Furthermore if α, γ label A, B, respectively, and (ϕ, ψ) label a corresponding element in the maximal torus, then $\alpha+\gamma = 2\phi$, $\alpha-\gamma = 2\psi$. The (α, γ) labels appear in table 2 in the appendix.

Since SU(2)$\times$SU(2) is a direct product, its reps may be labelled by pairs (j_1, j_2) and expressed as $D^{j_1j_2} = D^{j_1} \otimes D^{j_2}$, an outer Kronecker product, where D^j is the $(2j+1)$-dimensional rep of SU(2). It is easy to check that $D^{j_1j_2} \equiv D^4_{m_1m_2}$ if and only if $m_1 = j_1+j_2$ and $m_2 = j_1-j_2$. We notice also that $D^{j_1j_2} = D^{j_10} \otimes D^{0j_2}$, an inner Kronecker product, hence to reduce $D^{j_1j_2} \downarrow \boldsymbol{I}$, it is sufficient to know $D^{j_10} \downarrow \boldsymbol{I}$ and $D^{0j_2} \downarrow \boldsymbol{I}$. Now the character value at (α, γ) of $D^{j0} \downarrow \boldsymbol{I}$ is $[\sin(j+\frac{1}{2})\alpha]/\sin\frac{1}{2}\alpha$ and this is precisely the value of the character of $\widetilde{D^j \downarrow \boldsymbol{I}}$ at $\theta = \alpha$. Also, the character value at (α, γ) of $D^{0j} \downarrow \boldsymbol{I}$ is $[\sin(j+\frac{1}{2})\gamma]/\sin\frac{1}{2}\gamma$ which is the value of the character $D^j \downarrow \boldsymbol{I}$ at $\theta = \gamma$. The final set of entries in table 3 are the (j_1, j_2) labels of the reps of SU(2)$\times$SU(2), and serve to tie together some of the results of the previous section.

The procedure to be followed for the reduction of $D^{j_1j_2} \downarrow \boldsymbol{I}$ is now clear, for it is merely necessary to take the inner Kronecker product of $\widetilde{D^{j_1} \downarrow \boldsymbol{I}}$ with $D^{j_2} \downarrow \boldsymbol{I}$, taking advantage of equations (2.2) to (2.5) with related text and also the Clebsch–Gordan series of reps of $\boldsymbol{I}$.

We turn now to the problem of symmetrizing the reps of SO(4). From an inspection of the appended tables, we see that, with one exception, we are led directly back to the SO(3) approach discussed in § 2. The exception is in $G = D^{\frac{1}{2}\frac{1}{2}} \downarrow \boldsymbol{I}$, which is not expressible in the form $D^{j0} \downarrow \boldsymbol{I}$ or $D^{0j} \downarrow \boldsymbol{I}$. Also, we note that $I_{5/2}$ can be expressed as $D^{\frac{1}{2}1} \downarrow \boldsymbol{I}$ and $D^{1\frac{1}{2}} \downarrow \boldsymbol{I}$, which are not of a simple SO(3) nature. In order to symmetrize these representations we need some results from Gard and Backhouse (1974), Backhouse and Gard (1974b) and Gard (1974) on the symmetrization of reps of SO(3) and SO(4). First

$$(D^{\frac{1}{2}j})^{[\nu]} = \bigoplus_{m \geqslant m'} \sigma([\nu];[\mu],[\mu'])D(m,m') \otimes (D^{0j})^{[\mu]} \otimes (D^{0j})^{[\mu']}, \tag{3.6}$$

where $[\mu] = (\mu_1, \mu_2)$, $[\mu'] = (\mu'_1, \mu'_2)$ are UIR of $\boldsymbol{S}_m, \boldsymbol{S}_{m'}$, respectively, for $m+m' = n$, and $\sigma([\nu];[\mu],[\mu'])$ is the frequency of $[\nu]$ in $[\mu] \odot [\mu'] = [\mu] \otimes [\mu'] \uparrow \boldsymbol{S}_n$, induced from $\boldsymbol{S}_m \times \boldsymbol{S}_{m'}$. Also,

$$D(m,m') = \begin{cases} D^{\frac{1}{2}(m-m')0} \ominus D^{\frac{1}{2}(m-m'-2)0}, & \text{for } m \geqslant m'+2 \\ D^{\frac{1}{2}0}, & \text{for } m = m'+1 \\ D^{00}, & \text{for } m = m'. \end{cases} \tag{3.7}$$

The cases $j = \frac{1}{2}, 1$ are of interest to us, and for them we need

$$(D^{0\frac{1}{2}})^{[\mu]} = D^{0\frac{1}{2}(\mu_1-\mu_2)}, \tag{3.8}$$

$$\begin{aligned}(D^{01})^{[\mu]} = (D^{01})^{(\mu_1)} \oplus (D^{01})^{(\mu_1-1)} \oplus \ldots \oplus (D^{01})^{(\mu_2)} \\ \ominus (D^{01})^{(\mu_1-\mu_2-1)} \ominus \ldots \ominus (D^{01})^{(1)} \ominus (D^{01})^{(0)},\end{aligned} \tag{3.9}$$

in which the direct subtractions are absent if $\mu_1 = \mu_2$, and

$$(D^{01})^{(\lambda)} = D^{0\lambda} \oplus D^{0\lambda-2} \oplus \ldots \oplus \begin{cases} D^{00} & \text{if } \lambda \text{ is even} \\ D^{01} & \text{if } \lambda \text{ is odd.} \end{cases} \tag{3.10}$$

Equation (3.6) has a complicated appearance, but it is in fact more easily handled than one based on (2.1), because the symmetric group involvement is via the outer rather than the inner Kronecker product.

This concludes our discussion of the SO(4) orientated rep theory of $\boldsymbol{I}$. However, it is perhaps interesting to note briefly that certain generalizations are possible. We have in mind the fact that the regular tetrahedron and its symmetry group have analogues in all dimensions, namely that the proper Euclidean symmetry group of the regular $(n+1)$-hedroid in E_n is isomorphic to $\boldsymbol{A}_{n+1}$ and the full Euclidean symmetry group is isomorphic to $\boldsymbol{S}_{n+1}$. These groups are crystallographic. For further discussion we refer to Coxeter (1948). As another realization of this result we note that the extended set of fundamental weights of the rank $n-1$ Lie group SU(n) form the vertices of a regular n-hedroid in E_{n-1}, the weight space of SU(n), and that the Weyl group is $\boldsymbol{S}_n$ (for example, see Speiser 1965). From these geometric observations we have an explicit embedding of $\boldsymbol{A}_{n+1}$ in SO(n) and $\boldsymbol{S}_{n+1}$ in O(n). Also, it is not hard to check that the restriction to $\boldsymbol{S}_{n+1}$ of the self-representation of O(n) is the n-dimensional rep$[n, 1]$, confirming a result proved purely algebraically by Butler and King (1973).

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Appendix

We display below in tabular form the simples characters of I and the relationships between the various parameters used in the text.

Table 1. Character table of I.

	$C_1(1)$	$C_2(12)$	$C_3(12)$	$C_4(15)$	$C_5(20)$
A	1	1	1	1	1
T_1	3	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$	-1	0
T_2	3	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$	-1	0
G	4	-1	-1	0	1
H	5	0	0	1	-1
$E_{1/2}$	2	$(1+\sqrt{5})/2$	$(-1+\sqrt{5})/2$	0	1
$E_{7/2}$	2	$(1-\sqrt{5})/2$	$-(1+\sqrt{5})/2$	0	1
$G_{3/2}$	4	1	-1	0	-1
$I_{5/2}$	6	-1	1	0	0

Table 2. Class parameters of I.

Class (order)	$C_1(1)$	$C_2(12)$	$C_3(12)$	$C_4(15)$	$C_5(20)$
θ	0	$2\pi/5$	$4\pi/5$	π	$2\pi/3$
(ϕ, ψ)	$(0, 0)$	$(2\pi/5, 4\pi/5)$	$(4\pi/5, 8\pi/5)$	$(0, \pi)$	$(0, 2\pi/3)$
(α, γ)	$(0, 0)$	$(6\pi/5, -2\pi/5)$	$(12\pi/5, -4\pi/5)$	$(\pi, -\pi)$	$(2\pi/3, -2\pi/3)$

(*a*) The parameters θ, (ϕ, ψ), (α, γ) correspond to class parameters in SO(3), SO(4), SU(2) × SU(2) respectively.
(*b*) The other rotation angles in the classes of I have been omitted.

Table 3. Representation parameters of I.

I	A	T_1	T_2	G	H	$E_{1/2}$	$E_{7/2}$	$G_{3/2}$	$I_{5/2}$
SO(3)	0	1	$\tilde{1}$	$\tilde{\frac{1}{2}}\otimes\frac{1}{2}$	2	$\frac{1}{2}$	$\tilde{\frac{1}{2}}$	$\frac{3}{2}$	$\frac{5}{2}$
SO(4)	$(0, 0)$	$(1, -1)$	$(1, 1)$	$(1, 0)$	$(2, -2)$	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{2}, \pm\frac{3}{2})$	$(\frac{3}{2}, \pm\frac{1}{2})(\frac{5}{2}, \pm\frac{5}{2})$
SU(2) × SU(2)	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(0, 2)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(\frac{3}{2}, 0)$ $(0, \frac{3}{2})$	$(1, \frac{1}{2})(\frac{1}{2}, 1)$ $(\frac{5}{2}, 0)(0, \frac{5}{2})$

(*a*) An entry j or $\tilde{j}$ in row 2 indicates that the rep $D^j \downarrow I$ or $\widetilde{D^j \downarrow I}$ is equivalent to the rep of I in the same column.
(*b*) Entries (m_1, m_2) and (j_1, j_2) in rows 3 and 4 refer to reps $D^4_{m_1 m_2}$ and $D^{j_1 j_2}$ of SO(4) and SU(2) × SU(2), respectively.

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